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On Bounds for Cohomological Hilbert Functions

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1. INTRODUCTION

Let M be a finitely generated, graded module over a graded K-algebra A. Recently M. Brodman [3–5] studied systematically upper bounds for cohomological Hilbert functions $t \mapsto \operatorname{rank}_K[H^i_m(M)]_t$ (cf. Sect. 2 for notations). The main purpose of this paper is to show how the results of Brodmann can be improved. Thus we summarize in Sect. 3 some crucial results of Brodmann's method. Section 4 is devoted to the key results of this paper. Then we apply our methods for bounding cohomological Hilbert functions. We restrict ourselves to the special case of projective varieties in order to show as clearly as possible how the previous results can be used and to simplify the description of the bounds. First we study the highest cohomological Hilbert function relating it to the usual Hilbert function of the finite set of points arising from hyperplane sections of the variety. This method was already used by Castelnuovo in order to bound the genus of a curve by its degree. Indeed we state our results in Sect. 5 explicitly only as a bound for $h^n(\mathcal{O}_V)$ where n is the dimension of the variety V. If the variety is smooth this number is the geometric genus of the projective variety. Thus we extend in Sect. 5 some results known to be true for curves to higher-dimensional varieties.

In Sect. 6 we put all the previous results together in order to deduce bounds for the second-highest cohomological Hilbert function. This is intended as a sample that shows how our methods can be used to bound all cohomological Hilbert functions (cf. also [5]). We obtain bounds depending only on a small set of parameters like degree, dimension, and embedding dimension of the variety. This is somewhat related to results of [15, 2], but note that these results depend essentially on the Buchsbaum property.

It is an aim of bounding cohomological Hilbert functions to obtain vanishing results, that is, to obtain bounds for Castelnuovo's regularity

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reg $V = \min\{m \in \mathbb{Z} : h^i(I_V(j)) = 0$ for all $i + j \geqslant m$ $(i \geqslant 1)\}$, where I_V is the ideal sheaf of V. The conjecture of Eisenbud [8] states reg $V \leqslant \deg V -$ codim $V + 1$. It is open in general but known to be true, for example, in the case of curves. Somewhat surprisingly our general methods provide a new proof of this conjecture in the case of curves of small degree.

2. Notations and Preliminary Results

Throughout this paper let A be a graded K-algebra, e.g., $A = R/\mathfrak{a}$, where $R = K[x_0, ..., x_r]$ is a polynomial ring over an arbitrary algebraically closed field K and $\mathfrak{a} \subseteq R$ is a homogeneous ideal. We denote by $\mathfrak{m}_A$ or simply $\mathfrak{m}$ the irrelevant ideal $\bigoplus_{t>0} A_t$ of A.

Let $M = \bigoplus_{t \in \mathbb{Z}} M_t$ be a graded A-module. The ith graded part of M is also denoted by $[M]_i$. We set $e(M) = \sup\{t \in \mathbb{Z} : [M]_t \neq 0\}$. Let j be an integer; then $M(j)$ denotes the graded A-module whose A-module structure is the same as that of M and whose grading is given by $[M(j)]_t = [M]_{j+t}$.

Now let M always be a graded, finitely generated A-module of (Krull-) dimension $n+1$. The ith local cohomology module of M with support in $\mathfrak{m}$, denoted by $H^i_{\mathfrak{m}}(M)$, is also a graded A-module. We set $[h^i(M)]_t = \operatorname{rank}_K [H^i_{\mathfrak{m}}(M)]_t$ and call $t \mapsto [h^{n+1}(M)]_t$ and $t \mapsto [h^n(M)]_t$ the highest and the second-highest cohomological Hilbert function of M, respectively. Moreover we define Castelnuovos regularity reg(M) by $\operatorname{reg}(M) = \max\{i + e(H^i_{\mathfrak{m}}(M)) : i \in \mathbb{Z}\}$.

Recall that a set $\{a_1, ..., a_m\}$ of elements of A is said to be a filter-regular sequence for M if

$$e((a_1, ..., a_{i-1})M :_M a_i/(a_1, ..., a_{i-1})M) < \infty$$

for $i = 1, ..., m$.

Note that for M a filter-regular sequence $\{l_0, ..., l_n\}$ of forms $\in [A]_1$ exists (see, for example, [16, Lemma 1]). We denote by $h_M(t) := \operatorname{rank}_K [M]_t$, $p_M(t)$, and $h_0(M)$ the Hilbert function, the Hilbert polynomial, and the $n!$-fold of the leading coefficient of $p_M(t)$, respectively.

2.1. Definition. Let T be a 1-dimensional, finitely generated, graded module. A numerical function $h: \mathbb{Z} \to \mathbb{Z}$ is said to be a lower estimation of h_T if

(i) $0 \leqslant h(t) \leqslant h_T(t)$ for all $t \in \mathbb{Z}$

(ii) $h(t) = h_0(T)$ for all $t \gg 0$.

Moreover we set $N_T + 1 := \min\{t \in \mathbb{Z} : h(t) = h_0(T)\}$.

The following result describes a connection between cohomological and usual Hilbert functions. It generalizes [18, Lemma 5].

2.2. LEMMA. *Let M be a finitely generated, graded A-module of dimension $n+1$ ($n \in \mathbb{N}$) and depth >0. Let $\{l_1, ..., l_n\}$ be a filter-regular sequence for M of elements of $[A]_1$. Put $M^0 := M$ and $M^i := M^{i-1}/l_i M^{i-1}/H^0_{\mathfrak{m}}(M^{i-1}/l_i M^{i-1})$ for $i = 1, ..., n$. Let h be a lower estimation of h_{M^n}; then, with $N = N_{M^n}$,*

$$[h^{n+1}_{\mathfrak{m}}(M)]_t \leqslant \begin{cases} 0 & \text{for} \quad t > N-n \\ \binom{N-t}{n} h_0(M) - \sum\limits_{j=t+n}^{N} \binom{j-t-1}{n-1} h(j) & \text{for} \quad t \leqslant N-n. \end{cases}$$

Proof. We induct on n. If $n=0$ the result follows from [19]. Let $n=1$. In this case we write simply l instead of l_1. The exact sequence

$$0 \longrightarrow M(-1) \xrightarrow{\ l\ } M \longrightarrow M/lM \longrightarrow 0$$

provides $h_M(t) - h_M(t-1) = h_{M/lM}(t) = h_{M^1}(t) + [h^0(M/lM)]_t$ and, because of depth $M > 0$, the long exact cohomology sequence

$$0 \longrightarrow H^0_m(M/lM) \longrightarrow H^1_m(M)(-1) \xrightarrow{\ l\ } H^1_m(M) \longrightarrow \cdots.$$

Moreover it follows from [19] that

$$h_M(t) - p_M(t) = -[h^1(M)]_t + [h^2(M)]_t.$$

Putting it all together we obtain

$$\begin{aligned} &[h^2_{\mathfrak{m}}(M)]_{t-1} - [h^2_{\mathfrak{m}}(M)]_t \\ &\quad = h_0(M) + h_M(t-1) - h_M(t) + [h^1(M)]_{t-1} - [h^1(M)]_t \\ &\quad = h_0(M) - h_{M^1}(t) - [h^0(M/lM)]_t + [h^1(M)]_{t-1} - [h^1(M)]_t \\ &\quad \leqslant h_0(M) - h_{M^1}(t) \\ &\quad \leqslant h_0(M) - h(t). \end{aligned}$$

Since $h_0(M) = h_0(M^1)$ and $[h^2(M)]_t = 0$ for all $t \gg 0$ we get the assertion in case $n = 1$.

Let $n > 1$. The long exact cohomology sequence of

$$0 \longrightarrow M(-1) \xrightarrow{\ l_1\ } M \longrightarrow M^1 \longrightarrow 0$$

provides our statement as in the proof of Lemma 5 in [18]. Q.E.D.

A variety $V \subseteq \mathbb{P}^r$ is always assumed to be irreducible, reduced, and nondegenerated, i.e., V is not contained in a proper linear subspace of $\mathbb{P}^r$. A curve is a variety of dimension 1.

We define for integers a and $b \neq 0$: $\lceil a/b \rceil = \min\{t \in \mathbb{Z} : bt \geqslant a\}$, $\lfloor a/b \rfloor = \max\{t \in \mathbb{Z} : bt \leqslant a\}$.

3. Linear Systems of Hyperplane Sections

The results in this section describe a key observation needed for the results in [3–5]. Although not explicitly stated they are implicitly used in these papers. We need them later.

3.1. Proposition. *Let M be a finitely generated, graded A-module and $l_0, ..., l_s \in [A]_1$ elements such that all $0 \neq l \in Z := \{\alpha_0 l_0 + \cdots + \alpha_s l_s : \alpha_0, ..., \alpha_s \in K\}$ are filter-regular for M. We set $e_i = \sup\{e(H^i_{\mathfrak{m}}(M/lM)) : 0 \neq l \in Z\}$ $(i \geqslant 0)$. Then*

$$[h^i(M)]_{t+1} \leqslant \max\{0, [h^i(M)]_t - s\} \qquad \textit{for all} \quad t \geqslant e_i.$$

Proof. Let $0 \neq l \in Z$ and consider the exact sequence

$$0 \longrightarrow M/0 : l(-1) \xrightarrow{\ l\ } M \longrightarrow M/lM \longrightarrow 0.$$

Since l is filter-regular the long exact cohomology sequence provides for all $i \geqslant 0$ an exact sequence

$$H^i_{\mathfrak{m}}(M)(-1) \xrightarrow{\ l\ } H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(M/lM).$$

Now [4, Lemma 3] yields the assertion in view of the definition of e_i.

Q.E.D.

3.2. Corollary. *Let M be a finitely generated, graded A-module of dimension $n+1$ $(n \in \mathbb{N})$; then*

$$[h^{n+1}(M)]_{t+1} \leqslant \max\{0, [h^{n+1}(M)]_t - n\} \qquad \textit{for all} \quad t \in \mathbb{Z}.$$

Proof. According to [16, Lemma 1] there is a filter-regular sequence $\{l_0, ..., l_n\}$ of elements of $[A]_1$. Then all $l \in Z := \{\alpha_0 l_0 + \cdots + \alpha_n l_n : \alpha_0, ..., \alpha_n \in K\}$ distinct from zero are filter-regular for M. Therefore M/lM is n-dimensional and thus $H^{n+1}_{\mathfrak{m}}(M/lM) = 0$ for all $l \in Z - \{0\}$. Now Proposition 3.1 applies.

Q.E.D.

3.3. Corollary. *Let $A = R/I(V)$ be the coordinate ring of a variety $V \subseteq \mathbb{P}^r$. We set $e_i = \sup\{e(H^i_{\mathfrak{m}}(A/lA)) : l \in [A]_1 - \{0\}\}$. Then*

$$[h^i(A)]_{t+1} \leqslant \max\{0, h^i(A)]_t - r\} \qquad \textit{for all} \quad t \geqslant e_i.$$

Proof. Since V is nondegenerated all $l \in [A]_1$ distinct from zero are non-zero divisors of A. Thus Proposition 3.1 provides the assertion.

Q.E.D.

4. Single Hyperplane Sections

The results in this section are influenced by Chiarli [6]. They allow us to improve results of Brodmann (cf. Sect. 6).

4.1. Proposition. *Let M be a finitely generated, graded A-module and let $\{l_1, l_2\}$ be a filter-regular sequence for M of elements of $[A]_1$. Then for $1 \leqslant j \leqslant \dim M - 1$ and $t \geqslant e(H^{j-1}_{\mathfrak{m}}(M/(l_1, l_2)M)) - 1$ we have*

$$[h^j(M)]_{t+1} \leqslant \max\{0, [h^j(M)]_t - 1\}.$$

Proof. We have the following exact sequence for $j \geqslant 1$:

$$[H^{j-1}_{\mathfrak{m}}(M)]_t \xrightarrow{\beta_t} [H^{j-1}_{\mathfrak{m}}(M/l_1M)]_t \longrightarrow [H^j_{\mathfrak{m}}(M)]_{t-1} \longrightarrow [H^j_{\mathfrak{m}}(M)]_t \longrightarrow [H^j_{\mathfrak{m}}(M/l_1M)]_t. \tag{1}$$

We consider the following commutative diagram with exact rows:

$$\begin{array}{ccc}
[H^{j-1}_{\mathfrak{m}}(M)]_t & \xrightarrow{\beta_t} & [H^{j-1}_{\mathfrak{m}}(M/l_1M)]_t \\
\downarrow & & \downarrow \\
[H^{j-1}_{\mathfrak{m}}(M)]_{t+1} & \xrightarrow{\beta_{t+1}} & [H^{j-1}_{\mathfrak{m}}(M/l_1M)]_{t+1} \\
\downarrow & & \downarrow \\
[H^{j-1}_{\mathfrak{m}}(M/l_2M)]_{t+1} & \longrightarrow & [H^{j-1}_{\mathfrak{m}}(M/(l_1, l_2)M)]_{t+1}.
\end{array}$$

Since $[H^{j-1}_{\mathfrak{m}}(M/(l_1, l_2)M)]_{t+1} = 0$ for $t \geqslant f = e(H^{j-1}_{\mathfrak{m}}(M/(l_1, l_2)M))$ the above diagram shows that the surjectivity of β_t implies the surjectivity of β_{t+1} for $t \geqslant f$. Let $g = e(H^j_{\mathfrak{m}}(M))$. Then it follows from the sequence (1) that β_{g+1} is not surjective. Therefore β_t is not surjective for $f \leqslant t \leqslant g+1$. According to [16, Lemma 2] we have $e(H^j_{\mathfrak{m}}(M/l_1M)) < f$. Thus the exact sequence (1) shows for $f \leqslant t \leqslant g+1$

$$[h^j(M)]_{t-1} \geqslant [h^j(M)]_t + 1. \qquad \text{Q.E.D.}$$

In our applications we are not able to use Proposition 4.1 in order to estimate $[h^1(M)]_t$. Therefore we need

4.2. PROPOSITION. *Let M be a finitely generated, graded A-module of dimension $\geqslant 2$. Assume depth $M > 0$. Then we have for every filter-regular element $l \in [A]_1$ for M:*

$$[h^1_{\mathfrak{m}}(M)]_{t+1} \leqslant \max\{0, [h^1_{\mathfrak{m}}(M)]_t - 1\} \qquad \textit{for} \quad t \geqslant \operatorname{reg}(M/lM/H^0_{\mathfrak{m}}(M/lM)).$$

Proof. Put $M' = M/lM$ and $e = \operatorname{reg}(M'/H^0_{\mathfrak{m}}(M'))$. We get the following exact sequence because of depth $M > 0$,

$$0 \longrightarrow [H^0_{\mathfrak{m}}(M')]_t \longrightarrow [H^1_{\mathfrak{m}}(M)]_{t-1} \xrightarrow{\alpha_t} [H^1_{\mathfrak{m}}(M)]_t \longrightarrow [H^1_{\mathfrak{m}}(M')]_t. \tag{2}$$

Assume $[H^0_{\mathfrak{m}}(M')]_j = 0$ for some $j > e$. Then it follows $\operatorname{reg}(M') < j$ and thus $[H^0_{\mathfrak{m}}(M')]_t = 0$ for all $t \geqslant j$ (see, for example, [16, Theorem 1]). Therefore the sequence (2) shows that the injectivity of α_t implies that of α_{t+1} if $t > e$. Let $a = e(H^1_{\mathfrak{m}}(M))$. Then α_{a+1} is not injective by (2). Consequently α_t is not injective for $e + 1 \leqslant t \leqslant a + 1$. Furthermore it follows from (2) that α_t is surjective for $t \geqslant e$. Therefore we get

$$[h^1(M)]_t \leqslant [h^1(M)]_{t-1} - 1 \qquad \text{for} \quad e + 1 \leqslant t \leqslant a + 1. \qquad \text{Q.E.D.}$$

The case of 1-dimensional algebras is easy to handle.

4.3. LEMMA. *Let A be a 1-dimensional, graded K-algebra. Then we have for $t \geqslant 0$*

$$[h^1_{\mathfrak{m}}(A)]_t \leqslant \max\{0, [h^1_{\mathfrak{m}}(A)]_{t-1} - 1\}.$$

Proof. We may suppose A to be Cohen–Macaulay and we find a filter-regular element $l \in [A]_1$. The exact sequence

$$0 \to H^0_{\mathfrak{m}}(A/lA) \to H^1_{\mathfrak{m}}(A)(-1) \to H^1_{\mathfrak{m}}(A) \to 0$$

and [16, Theorem 1] provide the assertion. Q.E.D.

5. UPPER BOUNDS FOR THE HIGHEST COHOMOLOGICAL HILBERT FUNCTION OF PROJECTIVE VARIETIES

We state our results in this section only as an estimation of $[h^{n+1}(A)]_0$ because this simplifies their description and the proofs make it clear how one can derive bounds for $[h^{n+1}(A)]_t$ for all t using Lemma 2.2 or Corollary 3.3.

We need the following result of Ballico [1] (see also [17]).

5.1. LEMMA. *Let $V \subseteq \mathbb{P}^r$ be a variety of dimension n and degree d and let X be the intersection of V with a general linear subspace $L \subseteq \mathbb{P}^r$ of dimension $r-n$. Then X is a set of d points (in linear semi-uniform position in L) and the Hilbert function of X satisfies*

$$h_X(t) \geqslant \min\{d, t(r-n)+1\} \qquad (t \in \mathbb{N}).$$

5.2. *Remark.* A set $X \subseteq \mathbb{P}^c$ of d points is said to be in linear semi-uniform position if X spans $\mathbb{P}^c$ and every two linear subspaces of $\mathbb{P}^c$ of the same dimension and both spanned by points of X contain the same number of points of X.

A combination of the above lemma and Lemma 2.2 gives a result first proved by Harris [12] in case of char $K=0$ (see also [18]) and already mentioned in [1].

5.3. PROPOSITION. *Let $A = R/I(V)$ be the coordinate ring of a variety $V \subseteq \mathbb{P}^r$ of dimension n and degree d and let N and ε be integers defined by $d-1 = N(r-n)+\varepsilon$ and $0 < \varepsilon \leqslant r-n$. Then*

$$[h^{n+1}(A)]_0 \leqslant \binom{N}{n+1}(r-n) + \binom{N}{n}\varepsilon.$$

Proof. We use the notations of Lemma 2.2. According to Lemma 5.1 we may take $h(t) := \min\{d, t(r-n)+1\}$ for $t \geqslant 0$. Note that our N coincides with the integer N_{A^n} determined by this lower estimation h in the sense of Definition 2.1. Thus Lemma 2.2 yields the assertion. Q.E.D.

We now improve the above bound for varieties of codimension 2 generalizing Theorem 4 of [18] (for improvements in other cases see, for example, [7, 18]). A subscheme of $\mathbb{P}^r$ is said to be a complete intersection of type (a, b) if it is the complete intersection of two hypersurfaces of degree a and b.

5.4. THEOREM. *Let $V \subseteq \mathbb{P}^r_K$ (char $K=0$) be a variety of codimension 2 and degree d. Assume that V is not contained in a hypersurface of degree $<k$ $(k \geqslant 2)$ and $d > k(k-1)$. Then*

$$[h^{r-1}(A)]_0 \leqslant \binom{k+f-2}{r} + \binom{k+f-\varepsilon-2}{r-1} - \binom{k-1}{r} - \binom{f-1}{r} =: F(d, k),$$

where $A = R/I(V)$ is the coordinate ring of V and f, ε are defined by $d+\varepsilon = kf$ and $0 \leqslant \varepsilon < k$.

Moreover, if V is linked to a complete intersection of type $(1, \varepsilon)$ by a complete intersection of type (k, f) then $[h^{r-1}(R/I(V))]_0 = F(d, k)$.

Proof. In view of [18, Theorem 4] we must show only the first assertion. Consider the intersection X of V with a general linear subspace $L \subseteq \mathbb{P}^r$ of dimension 2. Then X is a set of d points in uniform position in $L \cong \mathbb{P}^2$ [11, p. 197]. We put $a = \min\{t \in \mathbb{N} : [I(X)]_t \neq 0\}$ and consider two cases.

Case 1. $a(a-1) < d$. Then we can use the proof of Theorem 4 in [18] and get $[h^{n+1}(A)]_0 \leqslant F(d, a)$.

Case 2. $a(a-1) \geqslant d$. Let B be the coordinate ring of X. We set $c_i = h_B(i) - h_B(i-1)$. Taking $h = h_B$ as a lower estimation of $h_{A''} = h_B$ it follows from Lemma 2.2 for $t \gg 0$ that

$$[h^{n+1}(A)]_0 \leqslant \sum_{i=r-1}^{t} \binom{i-1}{r-2} c_i. \tag{3}$$

The integers c_i satisfy the conditions

(i) $$\sum_{i=0}^{t} c_i = d \qquad \text{for} \quad t \gg 0$$

(ii) $$c_i = \begin{cases} i+1 & \text{for} \quad 0 \leqslant i < a \\ a & \text{for} \quad a \leqslant i < b \\ \leqslant \max\{0, c_{i-1} - 1\} & \text{for} \quad b \leqslant i, \end{cases}$$

where $b \geqslant a$ is an integer such that $d \leqslant ab$ [14, Proposition 1 and Corollary 2].

Claim.

$$\sum_{i=r-1}^{t} \binom{i-1}{r-2} c_i \leqslant \sum_{i=r-1}^{t} \binom{i-1}{r-2} \bar{c}_i =: S(d, a)$$

for all c_i's satisfying (i), (ii) and $t \gg 0$ where

$$\bar{c}_i = \begin{cases} i+1 & \text{for} \quad 0 \leqslant i < a \\ a+v+1-i & \text{for} \quad a \leqslant i < a+\delta-1 \\ a+v-i & \text{for} \quad a+\delta \leqslant i \leqslant a+v \\ 0 & \text{for} \quad a+v \leqslant i \end{cases}$$

and δ, v are defined by $d = \binom{a+1}{2} + \binom{v+1}{2} + \delta$ and $0 \leqslant \delta \leqslant v$.

Proof of the claim. Note that the $\bar{c}_i$'s also satisfy (i) and (ii). Thus assuming first $c_i \leqslant \bar{c}_i$ for all i we obtain $c_i = \bar{c}_i$ for all i by condition (i).

Now let $j \geqslant a$ be the first integer such that $c_j > \bar{c}_j$. Then there is a first integer $k > j$ such that $c_k < \bar{c}_k$. We define new integers d_i by $d_i = c_i$ if $i \neq j$ and $i \neq k$ and $d_j = c_j - 1$, $d_k = c_k + 1$. The new d_i's also satisfy the above conditions (i), (ii) and

$$\sum_{i=r-1}^{t} \binom{i-1}{r-2} c_i \leqslant \sum_{i=r-1}^{t} \binom{i-1}{r-2} d_i.$$

If one of the d_i's is again $> \bar{c}_j$ for some j we repeat this procedure with $c_i = d_i$. It stops if $d_i = \bar{c}_i$ for all i. Hence we get our claim.

The claim and (3) show in case 2

$$[h^{n+1}(A)]_0 \leqslant S(d, a).$$

The function

$$R(d, t) := \begin{cases} F(d, t) & \text{if} \quad t(t-1) < d \\ S(d, t) & \text{if} \quad t(t-1) \geqslant d \end{cases}$$

($t \geqslant 2$) is decreasing in t. According to [20] we know $a \geqslant k$, proving the first assertion because we have shown $[h^{n+1}(A)]_0 \leqslant R(d, a)$. Q.E.D.

5.5. *Remark.* In case of curves in $\mathbb{P}^3$ the above theorem yields Theorem 3.1 of [10]. In the proof we have used ideas from [11].

6. Bounds for the Second-Highest Cohomological Hilbert Function

Now we put all results of the previous sections together in order to obtain bounds depending only on a small set of parameters.

6.1. Theorem. *Let* $V \subseteq \mathbb{P}^r_K$ (char $K = 0$) *be a smooth variety of dimension* $n \geqslant 2$ *and degree* d *and let* $X = \operatorname{Proj}(C) \subseteq \mathbb{P}^{r-n}$ *be the intersection of* V *and* n *general hyperplanes* $H_1, ..., H_n$. *If* h *is a lower estimation for the Hilbert function of* X *we obtain with* $A = R/I(V)$

$$[h^n(A)]_t \leqslant \begin{cases} 0 & \textit{for} \quad t < 0 \\ \left[\binom{N+1}{n} - \binom{N-t}{n}\right] d + \sum_{j=t+n}^{N} \binom{j-t-1}{n-1} h(j) & \\ \quad - \sum_{j=n-1}^{N} \binom{j}{n-1} h(j) - \delta r & \textit{for} \quad 0 \leqslant t \leqslant N-n+1 \\ \max\{0, [h^n(A)]_{t-1} - 1\} & \textit{for} \quad N-n+2 \leqslant t < \lceil p/(n-1) \rceil \\ \max\{0, [h^n(A)]_{t-1} - r\} & \textit{for} \quad \lceil p/(n-1) \rceil \leqslant t, \end{cases}$$

where $N = N_C$ is defined by h in the sense of Definition 2.1 and

$$\delta = \begin{cases} 1 & \textit{if } [h^{n+1}(A)]_0 \neq 0 \\ 0 & \textit{otherwise} \end{cases} \qquad \textit{and} \qquad p = \binom{N}{n-1} d - \sum_{j=n-1}^{N} \binom{j-1}{n-2} h(j).$$

Proof. The Kodaira vanishing theorem [13] gives the assertion for $t<0$. Thus we may suppose $t \geq 0$. Let l_1, l_2 be the defining equations of H_1 and H_2, respectively. We set $A^1 = A/l_1A$ and $A^2 = A^1/l_2A^1$. Let $l \in [A]_1$ distinct from zero and $B = A/lA$. We consider the exact sequence

$$[H^n_{\mathfrak{m}}(A)]_{t-1} \xrightarrow{l} [H^n_{\mathfrak{m}}(A)]_t \longrightarrow [H^n_{\mathfrak{m}}(B)]_t \longrightarrow [H^{n+1}_{\mathfrak{m}}(A)]_{t-1}$$
$$\xrightarrow{l} [H^{n+1}_{\mathfrak{m}}(A)]_t \longrightarrow 0. \tag{4}$$

It provides for $t=0$

$$[h^n(B)]_0 = [h^n(A)]_0 - ([h^{n+1}(A)]_0 - [h^{n+1}(A)]_{-1}).$$

Thus $[h^n(B)]_0$ does not depend on l and Corollary 3.3 gives

$$[h^n(A)]_0 \leq [h^n(A^1)]_0 - \delta \cdot r.$$

Moreover (4) shows

$$[h^n(A)]_t \leq [h^n(A)]_{t-1} + [h^n(B)]_t. \tag{5}$$

Consequently we obtain because of $[h^n(A)]_{-1} = 0$ for $t \geq 0$

$$[h^n(A)]_t \leq \sum_{i=0}^{t} [h^n(A^1)]_i - \delta r, \tag{6}$$

showing the assertion for $0 \leq N-n+1$ since Lemma 2.2 applies to $[h^n(A^1)]_i = [h^n(A^1/H^0_{\mathfrak{m}}(A^1))]_i$.

It follows from Lemma 2.2 that $e(H^{n-1}_{\mathfrak{m}}(A^2)) \leq N-n+2$. Therefore Proposition 4.1 yields our statement for $t \geq N-n+2$.

Since $[h^n(A/lA)]_0 = [h^n(A^1)]_0$ for all $0 \neq l \in [A]_1$ we obtain from Corollary 3.2

$$[h^n(A/lA)]_t = 0 \qquad \text{for all} \quad t \geq \left\lceil \frac{[h^n(A^1)]_0}{n-1} \right\rceil.$$

According to Lemma 2.2 we know $[h^n(A^1)]_0 \leq p$. Thus we have in (4) for all $0 \neq l$ an epimorphism $[H^n_{\mathfrak{m}}(A)]_{t-1} \xrightarrow{l} [H^n_{\mathfrak{m}}(A)]_t$ if $t \geq \lceil p/(n-1) \rceil$. Applying [4, Lemma 3] we are done. Q.E.D.

The above theorem generalizes and improves Corollary 9 of [18].

If we replace the assumptions on the smoothness of V and the characteristic of K by the condition $[h^n(A)]_{-1}=0$ we obtain the same bounds as in the theorem for $t\geqslant 0$. More generally, knowing $[h^n(A)]_k=0$ for some integer k it is possible to deduce bounds of $[h^n(A)]_t$ for all $t\geqslant k$ as in the above proof. This last assumption makes sense because $[h^n(A)]_t=0$ for all $t\leqslant 0$ iff $H^n_{\mathfrak{m}}(A)$ is of finite length.

According to Lemma 5.1 we may choose $h(t)=\min\{d, t(r-n)+1\}$ $(t\geqslant 0)$ in Theorem 6.1. Then it provides bounds depending only on d, r, and n.

The next statement is similar to Theorem 6.1 but has a closer relation to the point of view of Brodmann. It generalizes and improves Proposition (A) in [4].

6.2. THEOREM. *With the assumption and notations of Theorem* 6.1 *we have*

$$[h^n(A)]_t \leqslant \begin{cases} 0 & \text{for} \quad t\leqslant 0 \\ (t+1)p-\delta r-\binom{t+1}{2}(r-1) & \text{for} \quad 0\leqslant t\leqslant \lfloor p/(r-1)\rfloor \\ [h^n(A)]_{t-1} & \text{for} \quad \lfloor p/(r-1)\rfloor < t\leqslant \lceil (d-1)/(r-n)\rceil - n \\ \max\{0, [h^n(A)]_{t-1}-1\} & \text{for} \quad \lceil (d-1)/(r-n)\rceil - n < t < \lceil p/(n-1)\rceil \\ \max\{0, [h^n(A)]_{t-1}-r\} & \text{for} \quad \lceil p/(n-1)\rceil \leqslant t, \end{cases}$$

where $p=[h^n(A/l_1A)]_0$ *and*

$$\delta=\begin{cases} 1 & \text{if} \quad [h^{n+1}(A)]_0\neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is similar to that of Theorem 6.1. We mention only the differences. We use Corollary 3.3 instead of Lemma 2.2 in order to deduce bounds from the relation (6) in the previous proof. This gives us for $t\geqslant 0$

$$[h^n(A)]_t\leqslant p-\delta r+\sum_{i=1}^{t} \max\{0, p-i(r-1)\},$$

providing together with relation (5) in the proof of Theorem 6.1 our assertion in case $t\leqslant\lceil (d-1)/(r-n)\rceil - n$.

According to Lemma 5.1 we may take, for $j\geqslant 0$, $h(j)=$

$\min\{d, j(r-n)+1\}$. Thus $N < \lceil (d-1)/(r-n) \rceil$ and our statement follows from Theorem 6.1. Q.E.D.

In the previous statements of this section we excluded curves. We have the following for curves, possibly singular.

6.3. THEOREM. *Let $C \subseteq \mathbb{P}^r$ be a curve of degree d with coordinate ring A. Let $H \subseteq \mathbb{P}^r$ be a general hyperplane and h be a lower estimation of the Hilbert function of $X = V \cap H$, N defined by h. Then*

$$[h^1(A)]_t \leqslant \begin{cases} 0 & \textit{for} \quad t \leqslant 0 \\ td - \sum_{i=1}^{t} h(i) & \textit{for} \quad 1 \leqslant t \\ \max\{0, [h^1(A)]_{t-1} - 1\} & \textit{for} \quad N+2 \leqslant t \\ \max\{0, [h^1(A)]_{t-1} - r\} & \textit{for} \quad d-r+1 \leqslant t. \end{cases}$$

Proof. The assertion is well-known in case $t \leqslant 0$ [19]. We have for all $0 \neq l \in [A]_1$ an exact sequence

$$\begin{aligned} [H^1_{\mathfrak{m}}(A)]_{t-1} \xrightarrow{l} [H^1_{\mathfrak{m}}(A)]_t \longrightarrow [H^1_{\mathfrak{m}}(A/lA)]_t \longrightarrow [H^2_{\mathfrak{m}}(A)]_{t-1} \\ \longrightarrow [H^2_{\mathfrak{m}}(A)]_t \longrightarrow 0. \end{aligned} \tag{7}$$

Putting $t = 1$ shows that $[h^1(A/lA)]_1$ does not depend on l. Thus we get

$$[h^1(A/lA)]_1 = d - r \qquad \text{for all} \quad 0 \neq l \in [A]_1. \tag{8}$$

This follows if we take $l = l'$, where l' is the defining form of H.

We obtain the following from (7) for $t \geqslant 1$ with $l = l'$ similar to (6) in the proof of Theorem 6.1:

$$[h^1(A)]_t \leqslant \sum_{i=1}^{t} [h^1(A/l'A)]_i \leqslant td - \sum_{i=1}^{t} h(i).$$

According to Lemma 2.2 we know $\operatorname{reg}(A/l'A/H^0_{\mathfrak{m}}(A/l'A)) = 1 + e(H^1_{\mathfrak{m}}(A/l'A)) \leqslant N+1$, providing our statement in the range $N+2 \leqslant t < d-r+1$ in view of Proposition 4.2.

Lemma 4.3 and (8) show for all $l \in [A]_1$ distinct from zero

$$e(H^1_{\mathfrak{m}}(A/lA)) \leqslant d - r.$$

Thus our statement follows for $t \geqslant d-r+1$ from (7) and [4, Lemma 3].

Q.E.D.

6.4. COROLLARY. *Let $C \subseteq \mathbb{P}^r$ be a curve of degree $d \leqslant 2r-1$ with coordinate ring A. Then*

$$\text{(i)} \qquad [h^1(A)]_t \leqslant \begin{cases} 0 & \textit{for} \quad t \leqslant 0 \\ d-r & \textit{for} \quad t = 1, 2 \\ d-r+2-t & \textit{for} \quad 3 \leqslant t \leqslant d-r \\ 0 & \textit{for} \quad d-r < t, \end{cases}$$

$$\text{(ii)} \qquad \operatorname{reg}(A) \leqslant \deg C - \operatorname{codim} C. \bullet$$

Proof. In Theorem 6.3 we may take due to Lemma 5.1

$$h(t) = \begin{cases} 1 & \text{for} \quad t = 0 \\ r & \text{for} \quad t = 1 \\ d & \text{for} \quad t > 1. \end{cases}$$

Therefore Theorem 6.3 shows (i). Assertion (ii) follows from (i) and Lemma 2.2. Q.E.D.

The conclusion of Corollary 6.4 (ii) is known to be true for arbitrary curves [9]. Statement (i) is new.

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